Involution on quiver varieties and quantum symmetric pairs

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§1. Quiver varieties

$\Gamma \subset SU(2)$ finite subgroup $\longleftrightarrow$ ADE classification

ρ_i : irreducible representation $\overset{\longleftrightarrow}{\text{McKay}}$ vertex of affine Dynkin diagram

$\longleftrightarrow$ $\mathfrak{g} \equiv \mathfrak{g}_\Gamma$: complex simple Lie algebra

e.g. $\Gamma = \mathbb{Z}/n \longleftrightarrow \mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$

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quiver variety

$\mathfrak{M} \equiv \mathfrak{M}(v, w)$: framed moduli space of Γ-equivariant torsion free sheaves E over $\mathbb{P}^2 = \mathbb{C}^2 \cup l_\infty$

v, w : (isomorphism classes of) representations of Γ

$E|_{l_\infty} \cong \mathcal{O}_{l_\infty} \otimes \bigoplus \rho_i^{\oplus w_i}$

framing

$H^1(E(-l_\infty)) \cong \bigoplus \rho_i^{\oplus v_i}$

Assume $w_0 = 0 = v_0$

So finite Dynkin diagram

Fact. smooth quasi-projective variety

§2. Quiver varieties and representation theory

[N '94] topology of $\mathfrak{M}$ $\rightsquigarrow$ representations of $\mathfrak{g}$

$\bigoplus_{v} H_{\text{middle}}(\mathfrak{M}(v,w))$ is an irreducible finite dimensional representation of $\mathfrak{g}$ with highest weight $= w$

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[N, Varagnolo '98] $\mathbb{C}^* \times T_w \curvearrowright \mathfrak{M}$ action on base $(x,y) \mapsto (tx, ty)$ $(x,y) \in \mathbb{C}^2$

framing $T_w = \prod_i (\mathbb{C}^\times)^{w_i}$

$\bigoplus_v K_{equ}(\mathfrak{M}(v,w))$, $\bigoplus_v H^*_{equ}(\mathfrak{M}(v,w))$ are representations of

quantum loop algebra / Yangian associated with $\mathfrak{g}$

$U_q(L\mathfrak{g})$: deformation of $L\mathfrak{g} = \mathfrak{g}[z, z^{-1}]$ $\qquad Y_\hbar(\mathfrak{g})$: deformation of $C\mathfrak{g} = \mathfrak{g}[z]$

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finite dimensional

not semisimple

not commutative (monoidal)

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M, N : representations of $\mathfrak{g}$ $\Rightarrow$ $M \otimes N \cong N \otimes M$ as representations of $\mathfrak{g}$

$$m \otimes n \mapsto n \otimes m$$

It is no longer true for $Y_\hbar(\mathfrak{g})$.

What is true?

$\exists \{M(u)\}_{u \in \mathbb{C}}$: 1-parameter family of representations

(spectral parameter) (corresponding to $\mathfrak{g}[z] \to \mathfrak{g}[z]$, $z \mapsto z+u$)

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given by R-matrix, which is a rational function in u.

§3. Maulik-Okounkov stable envelope

Fix a decomposition $w = w^1 + w^2$ $\quad (w_i = w_i^1 + w_i^2)$

$$\mathbb{C}^* \ni (1 \quad , t)$$

$$\Longrightarrow \quad \mathcal{M}(v,w)^{\mathbb{C}^*} \cong \coprod_{v = v^1 + v^2} \mathcal{M}(v^1, w^1) \times \mathcal{M}(v^2, w^2)$$

$$\therefore \quad H^*_{equ}(\mathcal{M}(v,w)) \xrightarrow{\text{restriction}} \bigoplus_{v^1+v^2=v} H^*_{equ}(\mathcal{M}(v^1,w^1)) \otimes H^*_{equ}(\mathcal{M}(v^2,w^2))$$

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characterized uniquely by certain upper triangular properties.

- depending on order (w^1, w^2)
- choice of sign $\pm$ of $\sqrt{}$ of a certain bundle (polarization)
- isomorphism $\otimes \operatorname{Frac} H^*_{equ}(pt)$

$$\boxed{\begin{array}{l} (\text{Stab}^-)^{-1} \circ \text{Stab}^+ \qquad (+ : w^1, w^2, \ - : w^2, w^1) \\ \quad \text{gives the R-matrix of } Y_\hbar(\mathfrak{g}) \end{array}}$$

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gives the R-matrix of $Y_\hbar(\mathfrak{g})$

In fact, one can reconstruct $Y_\hbar(\mathfrak{g})$ from R-matrix. Thus stable envelope gives a different proof of [Varagnolo '98].

§4. Involution on quiver varieties and quantum symmetric pairs (after Y. Li)

Fact. quiver variety

$\mathfrak{M} \equiv \mathfrak{M}(v,w)$: framed moduli space of Γ-equivariant torsion free sheaves E over $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$

with $w_0 = 0 = v_0$ is a framed moduli space of vector bundles over $\widetilde{\mathbb{P}^2/\Gamma} = \widetilde{\mathbb{C}^2/\Gamma} \cup \ell_\infty/\Gamma$

$\Rightarrow \quad \mathfrak{M}(v,w) \xrightarrow{*} \mathfrak{M}(v^*, w^*)$ given by $E \mapsto E^*$ dual vector bundle

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Choose
- $V = V^*$ $\quad (\Leftrightarrow$ 1st Chern class $c_1(E) = 0)$
- $W = W^*$

a bit more precisely : choose $\bigoplus \rho_i^{\oplus w_i} \cong \left(\bigoplus \rho_i^{\oplus w_i}\right)^*$

symplectic or orthogonal form

We get an involution $\sigma \curvearrowright \mathfrak{M}(v,w)$

such that the fixed point locus $\mathfrak{M}(v,w)^\sigma$ is

a framed moduli space of Symplectic or Orthogonal bundles over $\widetilde{\mathbb{P}^2/\Gamma}$.

__Remark__. Li considers more general involutions $\sigma = (\underset{\text{auto}}{\text{diagram}}) \circ \text{dual}$

e.g. $\mathfrak{M}(v,w)^\sigma$ = cotangent bundle of flags $\mathbb{C}^N \supset V_1 \supset V_2 \supset \cdots \supset V_2^\perp \supset V_1^\perp \supset \{0\}$

Stable envelope

$\mathbb{C}^* \curvearrowright \mathfrak{M}(V,W)^\sigma$ must be compatible with symplectic/orthogonal form.

$\rightsquigarrow \quad W = \underset{\text{self dual}}{W^0} + W^1 + \underset{\text{dual}}{W^{-1}}$ instead of $W = W^1 + W^2$ before

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self dual $\circlearrowleft \sigma$, dual

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<u>Observation</u> $\quad \left(\mathfrak{M}(v,w)^\sigma\right)^{\mathbb{C}^*} \cong \coprod \mathfrak{M}(v^0,w^0)^\sigma \times \mathfrak{M}(v^1,w^1)$

usual quiver variety

$$H^*_{equ}(\mathfrak{M}(v,w)^\sigma) \underset{\mathrm{Stab}^\pm}{\overset{}{\rightleftarrows}} \bigoplus H^*_{equ}(\mathfrak{M}(v^0,w^0)^\sigma) \otimes H^*_{equ}(\mathfrak{M}(v^1,w^1))$$

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K-matrix

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<u>Observation</u> $\left(\mathfrak{M}(v,w)^\sigma\right)^{\mathbb{C}^*} \cong \coprod \mathfrak{M}(v^0,w^0)^\sigma \times \underbrace{\mathfrak{M}(v^1,w^1)}_{\text{usual quiver variety}}$

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K-matrix

Rep. of twisted Yangian $\longleftarrow$ Rep. of twisted Yangian $\otimes$ <u>Rep. of $Y_\hbar(\mathfrak{g})$</u>

$\uparrow$ module category of the monoidal category $\uparrow$

§5. Quantum symmetric pairs (very brief)

$\mathfrak{g}$: semisimple Lie algebra $/\mathbb{C}$

σ : involution

$(\mathfrak{g}, \mathfrak{g}^\sigma)$: symmetric pair $\quad (\mathfrak{sl}_n, \mathfrak{so}_n), (\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n})$

$(\mathfrak{sl}_n, \mathfrak{s}(\mathfrak{gl}_p \oplus \mathfrak{gl}_q)) \quad n = p+q$

$\mathfrak{g}[z] \circlearrowleft \sigma \otimes (-1) \; ; \; X(z) \longmapsto \sigma(X(-z))$ involution

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twisted Yangian $Y_\hbar^{tw}(\mathfrak{g}^\sigma) =$ quantization of $\mathfrak{g}[z]^{\sigma\otimes(-1)}$

different from Yangian of $\mathfrak{g}^\sigma$.

$Y_\hbar^{tw}(\mathfrak{g}^\sigma) \subset Y_\hbar(\mathfrak{g}) \qquad \Delta Y_\hbar^{tw}(\mathfrak{g}^\sigma) \subset Y_\hbar^{tw}(\mathfrak{g}^\sigma) \otimes Y_\hbar(\mathfrak{g})$

coideal subalgebra

$\underline{\mathrm{Rep}\, Y_\hbar^{tw}(\mathfrak{g}^\sigma)} \otimes \mathrm{Rep}\, Y_\hbar(\mathfrak{g}) \longrightarrow \mathrm{Rep}\, Y_\hbar^{tw}(\mathfrak{g}^\sigma)$

module category

Thm (Li'19) $\bigoplus_{v} H^{*}_{equ}(\mathfrak{M}(v,w)^{\sigma})$ is a representation of $Y^{tw}_{\hbar}(\mathfrak{g}^{\sigma})$ constructed by K-matrix.

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<u>Remark</u> [N.'26]: compute K-matrix explicitly in examples
$\Rightarrow Y^{tw}_\hbar(\mathfrak{g}^\sigma)$ = twisted Yangian in the literature
Olshanski, Molev, Ragoucy

[N'26] also pointed out an obstruction on polarization $\mathcal{I}$.
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more important remark:
This construction works only σ = (diagram auto) $\circ$ (Chevalley inv)
quasi-split type
e.g. $(\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n})$ is excluded.
$\longrightarrow$ Need to consider singular varieties

Singular moduli already treated
in Braverman–Finkelberg–N
"Instanton moduli spaces and W-algebras"

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Conj. (1) character formula in terms of intersection cohomology
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Thank you very much !